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Ronner, Arjen E; Sterken, Elmer

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GENERALIZATION OF $M(x)$ -MATRICES Application to Mathematical Economics and Econometrics

Arjen E. RONNER and Elmer STERKEN

University of Groningen, Groningen, The Netherlands

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Square matrices with properties such as non-negativity, diagonal dominance and equal row sums frequently occur in mathematical economics. In this paper Sierksma's matrices of class $M(x)$ are generalized to multisector models and are applied to the problem of pooling cross-section and time series data.

1. Introduction

Matrix theory is widely used in analyzing economic systems. Economical arguments often impose restrictions on the elements of matrices. Here we study a special class of matrices with some useful properties and applications. Input–output-modelling theory is one field of economical research with an increasing interest in matrix classes. The open Leontief model relates the final demand vector with the gross production vector

$$(I_n - T)y = c, \quad \text{where} \quad (1)$$

I_n is the $(n \times n)$ unit matrix,
 T is the $(n \times n)$ input-coefficient matrix,
 y is the gross production vector, and
 c is the final demand vector.

The impact of final demand on production in sectors depends on the sector-structure. Supposing the long-term sector-structure to be fixed, short-term changes in final demand will give rise to changes in gross

production. An important topic in this analysis of changes in demand and production is the formulation of conditions, which are sufficient to prove that a certain sector will gain relatively more from a relative stronger rise of final demand in this sector. In other words: if the change of the final demand in sector i exceeds the change in sector j , will the gross production react in a similar way?

Sierksma (1979) studied the following equivalence:

$$\Delta y_i < \Delta y_j \Leftrightarrow \Delta c_i < \Delta c_j, \quad i, j = 1, \dots, n, \quad i \neq j. \quad (2)$$

He proved that, under regularity conditions, (2) is equivalent to the assertion that T belongs to a certain class of matrices, called $M(x)$. Besides the open Leontief model this concept can be used in e.g., the Stolper–Samuelson framework

$$Aw = p, \quad \text{where} \quad (3)$$

A is the matrix of factor shares,
 p is the vector of product prices, and
 w is the vector of money rewards.

Wegge and Kemp (1969) derived necessary and sufficient conditions regarding the factor shares matrix A for the problem

$$\Delta p_i > 0, \quad \Delta p_j = 0 \Leftrightarrow \Delta w_i > 0, \quad \Delta w_j < 0, \quad i, j = 1, \dots, n, \quad i \neq j. \quad (4)$$

However, they did not provide a class of matrices, satisfying these conditions. Matrices of class $M(x)$ seem to be sufficient for (4), and necessary too, if statement (4) is put in a slightly more general framework like (2) [Sierksma, Steenge and Sterken (1983)].

In this paper we shall discuss a simultaneous vector approach in which elements of the matrix M are allowed to be $(p \times p)$ matrices. Section 3 provides an application of class $\mathcal{M}(n, p, M_{ii}, X)$ in mathematical economics and section 4 gives a useful application of symmetric matrices of class $\mathcal{M}(n, p, M_{ii}, X)$ in econometrics. First some definitions and basic theorems are presented.

Definition 1. A square $(n \times n)$ matrix $M = \{m_{ij}\}$ is said to belong to

class $M(x)$ if and only if $m_{jj} - m_{ij} = x$, $x \neq 0$, $i, j = 1, \dots, n$, $i \neq j$.

This definition implies that if $M \in M(x)$, then

$$M = \begin{bmatrix} m_{11} & m_{22} - x & \cdots & m_{nn} - x \\ m_{11} - x & m_{22} & \cdots & m_{nn} - x \\ \vdots & \vdots & & \vdots \\ m_{11} - x & m_{22} - x & \cdots & m_{nn} \end{bmatrix}.$$

In this paper a simultaneous sector approach is introduced in which the elements of the matrix M are allowed to be $(p \times p)$ matrices.

Definition 2. A square $(np \times np)$ matrix M is said to belong to class $\mathcal{M}(n, p, M_{ii}, X)$ if and only if M can be partitioned in $(n \times n)$ submatrices M_{ij} ($p \times p$) such that $M_{jj} - M_{ij} = X$, with $X \neq 0$, $i, j = 1, \dots, n$, $i \neq j$.

The following generalization of Sierksma's theorems 16 and 18 holds.

Theorem 1. Let $M \in \mathcal{M}(n, p, M_{ii}, X)$ and define $S = \sum_{i=1}^n M_{ii} - (n-1)X$. Let S and X be non-singular, then

$$(i) \quad M^{-1} = (I_n \otimes S^{-1}) \begin{bmatrix} X - M_{11} + S & \cdots & X - M_{nn} \\ \vdots & & \vdots \\ X - M_{11} & \cdots & X - M_{nn} + S \end{bmatrix} \\ \times (I_n \otimes X^{-1}),$$

or equivalently, $M^{-1} \in \mathcal{M}(n, p, S^{-1}(X - M_{ii} + S)X^{-1}, X^{-1})$.

$$(ii) \quad M = |X|^{n-1} |S|.$$

Proof. (i) Let

$$A = \sum_{i=1}^n s_n e'_i \otimes (M_{ii} - X), \quad \text{and}$$

$$B = \sum_{i=1}^n s_n e'_i \otimes (S^{-1}(X - M_{ii})X^{-1}), \quad (5)$$

with s_n the $n \times 1$ vector $(1, \dots, 1)'$ and e_i the i th unit vector. Note that $B = -(I_n \otimes S^{-1})A(I_n \otimes X^{-1})$ and $AB = (I_n \otimes (S - X))B$. Then

$$M = A + (I_n \otimes X) \quad \text{and} \quad M^{-1} = B + (I_n \otimes X^{-1}), \quad (6)$$

since $MM^{-1} = AB + A(I_n \otimes X^{-1}) + (I_n \otimes X)B + I_{np} = (I_n \otimes (S - X))B - (I_n \otimes S)B + (I_n \otimes X)B + I_{np} = I_{np}$. The proof of (ii) is immediate. Q.E.D.

In section 3 an application is given of the case $M_{ii} = \bar{M}$ for each $i = 1, \dots, n$. Therefore the following corollary is useful:

Corollary 1. If the $(np \times np)$ matrix $M \in \mathcal{M}(n, p, \bar{M}, X)$, then

$$M = J_n \otimes S + (I_n - J_n) \otimes X \quad \text{and} \quad M^{-1} = J_n \otimes S^{-1} + (I_n - J_n) \otimes X^{-1},$$

where

$$I_n = n^{-1} s_n s_n'.$$

2. On the application of class $\mathcal{M}(n, p, M_{ii}, X)$ in mathematical economics

Suppose that an np -sector input-output system can be divided into n subgroups of sectors, such that an $M(x)$ -structure holds over the subgroups.

Notation 1. We state that, if z_1 and z_2 are $(n \times 1)$ vectors $z_1 \ll z_2 \Leftrightarrow \forall i = 1, \dots, n, z_1^i < z_2^i$, where z_j^i is the i th element of z_j .

Definition 3. An $(n \times n)$ non-negative matrix T is called irreducible if $\forall i, j = 1, \dots, n \exists m = m(i, j) > 0$ such that $t_{ij}^{(m)} > 0$.

Lemma 1. If T is an $(n \times n)$ irreducible non-negative matrix and T has a dominant eigenvalue $\rho < \varphi$ then $(\varphi I - T)^{-1} \gg 0$.

Proof. See Debreu and Herstein (1953).

Note that if $T \in \mathcal{M}(n, 1, m_{ii}, x)$ with $\varphi > s > x > 0$, $(\varphi I - T)^{-1} \gg 0$.

The open Leontief model is characterized by $\varphi = 1$. Now we are able to give a multisector alternative of Sierksma's theorem 19.

Theorem 2. If $T \in \mathcal{M}(n, p, M_{ii}, X)$ with X a $(p \times p)$ non-negative, irreducible matrix with dominant eigenvalue $\rho < \varphi$, then

$$\Delta y_i \ll \Delta y_j \Leftrightarrow \Delta c_i \ll \Delta c_j, \quad i, j = 1, \dots, n, \quad i \neq j \quad (7)$$

holds.

Proof. $\Delta c_i \ll \Delta c_j \Leftrightarrow (\varphi I_p - X)\Delta y_i \ll (\varphi I_p - X)\Delta y_j$. With the properties of X $(\varphi I_p - X)^{-1} \gg 0$ holds according to Lemma 1. So $\Delta y_i \ll \Delta y_j$. Q.E.D.

Note that the opposite is not true, otherwise the equivalence of the assertions in Sierksma's theorem is violated.

The same concept can be used in the Stolper–Samuelson framework. If $A \in \mathcal{M}(n, p, M_{ii}, X)$ both $\Delta w_i \ll \Delta w_j \Leftrightarrow \Delta p_i \ll \Delta p_j$ and $\Delta p_i \gg 0, \Delta p_j = 0 \Leftrightarrow \Delta w_i \gg 0, \Delta w_j \ll 0$ hold, for each $i, j = 1, \dots, n; i \neq j$.

3. On the use of the class $\mathcal{M}(n, p, M_{ii}, X)$ in pooling cross-section and time series data

One of the most frequently used techniques in pooling cross-section and time series data is the analysis of variance. Suppose we have the following model:

$$y_{it} = \sum_{s=1}^k x_{its} \beta_{is} + u_{it}, \quad i = 1, \dots, n, \quad t = 1, \dots, T, \quad (8)$$

where y_{it} , x_{its} and u_{it} are $(p \times 1)$ -vectors and β_{is} is a scalar. This model may describe the sales of p firms over n regions which are explained by the marketing instruments x_{ijs} . Given the covariance matrix of the disturbances the parameters are usually estimated by maximum likelihood methods, cf. Wansbeek and Kapteyn (1982). Therefore we need the inverse of the covariance matrix. All – at least, known to us – error components covariance matrices have a mathematical structure that corresponds to class $\mathcal{M}(n, p, \bar{M}, X)$. Let us consider the two error

components model. We assume that the errors u_{it} (8) are decomposed as

$$u_{it} = \eta_t + \varepsilon_{it},$$

where η_t and ε_{it} are independently distributed as

$$\eta_t \sim N_p(0, \Gamma) \quad \text{and} \quad \varepsilon_{it} \sim N_p(0, \Delta),$$

with Γ positive semidefinite and Δ positive definite, both of order p . Define the $(p \times n)$ matrices and $(pn \times 1)$ vectors

$$V_t = (u_{1t}, \dots, u_{nt}) \quad \text{and} \quad u_t = \text{vec } V_t, \quad t = 1, \dots, T.$$

Then u_1, \dots, u_T are i.i.d. as $N_{np}(0, \Omega_1)$, with

$$\Omega_1 = \begin{bmatrix} \Gamma + \Delta & \cdots & \Gamma \\ \Gamma & & \vdots \\ \vdots & & \vdots \\ \Gamma & \cdots & \Gamma + \Delta \end{bmatrix}.$$

Note that $\Omega_1 \in \mathcal{M}(n, p, \Gamma + \Delta, \Delta)$, so according to Corollary 1 we can write $S_1 = n\Gamma + \Delta$ and

$$\Omega_1 = J_n \otimes S_1 + (I_n - J_n) \otimes \Delta, \quad (9)$$

$$\Omega_1^{-1} = J_n \otimes S_1^{-1} + (I_n - J_n) \otimes \Delta^{-1}. \quad (10)$$

The three error components model can be treated in a similar way.

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